

Parameter-space analysis of functional path-flow substitutions in Wardrop network equilibrium models

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Abstract

We study a path-flow Wardrop network equilibrium model in which total demand is split among paths subject to non-negativity, demand conservation, and a cost-balance condition. Each path flow is reparameterized as $q_p = \phi(s_p)$ for a scalar function ϕ and a parameter s_p , and we build a single mathematical framework for comparing six choices of ϕ : cosine, sine, the exponential, the hyperbolic sine, the hyperbolic cosine, and the shifted hyperbolic cosine $\phi(s) = \cosh(s) - 1$. For each substitution we prove a feasibility bound, identify the admissible domain on which it is injective, give its closed-form inverse, and derive the dimensionless condition number $\kappa_p = |\phi'(s_p) s_p / q_p|$. We prove — conditionally, not universally — that if two admissible substitutions produce the same flow vector q^* , then the link flow, costs, and all equilibrium quantities are identical; the substitutions differ in feasibility, in the zero-flow property (ability to represent an unused path at a finite parameter), in parameter uniqueness, and in κ_p . A fully worked example on a four-node, five-link communication network with a naturally unused path illustrates the theory with reproducible, independently verified computations.

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1. Introduction

1.1. Background

Wardrop equilibrium [1] is the canonical model for routing demand across multiple paths in a congested network: all used paths have equal cost, and no unused path has a lower cost. Dafermos [2] and Smith [3] established the variational-inequality formulation that underpins modern existence and uniqueness results, while Nagurney [4] provided a comprehensive treatment of network economics. Complementarity-problem formulations—which explicitly represent unused paths through a zero-flow condition—are

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developed in Facchinei & Pang [5]. In communication networks, user-equilibrium routing of data packets satisfies formally identical conditions: Kelly *et al.* [6] show that proportionally fair allocation coincides with a Wardrop-type equilibrium, and Roughgarden & Tardos [7] analyse the price of anarchy in such settings. Igbinosun & Udoenoh [8] recently applied Dijkstra-based flow optimisation to urban traffic networks; closely related flow-balance principles govern packet routing in communication systems. Channel-signal equalization [9–11] shares the demand-balancing motivation but operates at the signal level rather than the path-flow level.

In numerical methods for computing Wardrop equilibria, it is standard practice to introduce a smooth reparameterization $q_p = \phi(s_p)$ that maps an unconstrained parameter s_p to a non-negative flow q_p , eliminating the non-negativity constraint from the search space. Different choices of ϕ appear in the literature without systematic comparison. This paper provides that comparison for six specific functions, focusing on the mathematically precise properties that distinguish them: the zero-flow property, injectivity, feasibility range, and the dimensionless condition number $\kappa_p = |\phi'(s_p)s_p/q_p|$.

1.2. Contribution

Six substitutions are studied: $\cos(s)$, $\sin(s)$, e^s , $\sinh(s)$, $\cosh(s)$, and $\cosh(s) - 1$. The specific contributions are: (i) a restructured admissibility framework separating basic admissibility (Definition 3.2) from the zero-flow property (Definition 3.3); (ii) a strengthened feasibility proposition with explicit continuity and open-endpoint conditions (Proposition 3.1); (iii) a uniqueness proposition with proof (Proposition 3.2); (iv) a carefully scoped equilibrium-invariance theorem with an explicit remark on its conditional nature (Theorem 3.1); (v) a classification theorem with proof covering all six substitutions (Theorem 4.1); (vi) a corrected corollary addressing general path capacities (Corollary 6.1); and (vii) a four-node, five-link network example with a genuinely unused path and dimensionless condition numbers (Section 5).

1.3. Organisation

Section 2 fixes notation and recalls the Wardrop model. Section 3 presents the substitution framework, admissibility definitions, Propositions 3.1–3.2, and Theorem 3.1. Section 4 proves Theorem 4.1. Section 5 gives the network example. Section 6 summarises the comparative results, and Section 7 concludes.

2. Preliminaries: the Wardrop network equilibrium model

The network is a directed graph $D = (V, L)$ where V is the node set and L the link set. A directed link is $d = (i \rightarrow j)$, $d \in L$. Let W be the set of origin–destination (OD) pairs; for $w \in W$ let $b_w > 0$ be the demand on pair w , and let P_w be the set of admissible paths. The feasible set is

$$\sum_{p \in P_w} q_p = b_w, \quad \forall w \in W, \quad (1)$$

$$q_p \geq 0, \quad \forall p \in P_w, w \in W. \quad (2)$$

Eqs. (1) and (2) define the demand-conservation and non-negativity constraints, respectively. In matrix form, $b = Bq$, $q \geq 0$, where B is the OD/path incidence matrix. The link/path incidence matrix A has $\alpha_{dp} = 1$ if path p uses link d , and 0 otherwise. Link flows are $f = Aq$ and link costs are $t_d = T_d(f_d)$. Path costs are

$$g_p(q) = \sum_{d \in L} \alpha_{dp} t_d, \quad G(q) = A^T T(Aq). \quad (3)$$

Definition 2.1 (Wardrop equilibrium). A feasible flow q^* is a Wardrop equilibrium if, for every $p \in P_w$ and every $w \in W$, there exist costs u_w^* such that

$$g_p(q^*) \geq u_w^*, \quad q_p^*(g_p(q^*) - u_w^*) = 0. \quad (4)$$

In vector form: $G(q^*) \geq B^T u^*$ and $q^* \circ (G(q^*) - B^T u^*) = 0$.

An active path ($q_p^* > 0$) carries cost equal to u_w^* ; an unused path ($q_p^* = 0$) carries cost no lower than u_w^* .

3. Functional path-flow substitution

3.1. Definition and transformed model

Definition 3.1 (Functional substitution). A functional substitution is a scalar function ϕ together with a domain S_p for each path, such that $q_p = \phi(s_p)$ replaces the flow q_p by a parameter s_p .

The substitution transforms the feasible set to $\sum_{p \in P_w} \phi(s_p) = b_w$, $\phi(s_p) \geq 0$, and the equilibrium conditions to

$$g_{\phi,p}(s^*) \geq u_w^*, \quad \phi(s_p^*)(g_{\phi,p}(s^*) - u_w^*) = 0. \quad (5)$$

Note that Eq. (5) is written in terms of $\phi(s_p^*)$, not s_p^* , because equilibrium is a condition on actual flow.

3.2. Admissibility framework

Definition 3.2 (Basic admissibility — C1, C4, C5). A substitution ϕ is *admissible* for path p if:

(C1) *Non-negativity*: $\phi(s_p) \geq 0$ for all $s_p \in S_p$.

(C4) *Injectivity on S_p* : $\phi(s_1) = \phi(s_2) \Rightarrow s_1 = s_2$.

(C5) *Domain admissibility*: S_p is chosen so that $\phi(S_p) \subseteq [0, \infty)$.

Definition 3.3 (Zero-flow property — C3). An admissible substitution has the *zero-flow property* if there exists a finite $s_p^0 \in S_p$ with $\phi(s_p^0) = 0$.

The zero-flow property is essential when the equilibrium has unused paths ($q_p^* = 0$): without a finite parameter value representing zero flow, the complementarity condition Eq. (5) cannot be enforced at a finite s_p^* .

Definition 3.4 (Demand conservation — C2). A substitution satisfies C2 for OD pair w if the equation $\sum_{p \in P_w} \phi(s_p) = b_w$ is solvable for some $s \in \prod_{p \in P_w} S_p$.

Definition 3.5 (Sensitivity differentiability — C6). Where ϕ is differentiable and $\phi'(s_p) \neq 0$, the local parameter-to-flow sensitivity is $ds_p/dq_p = 1/\phi'(s_p)$. The dimensionless condition number is

$$\kappa_p = \left| \frac{\phi'(s_p) s_p}{\phi(s_p)} \right| = \left| \frac{d \ln q_p}{d \ln s_p} \right|. \quad (6)$$

The condition number in Eq. (6) is used below to compare the substitutions.

Lemma 3.1 (Scale-invariance of κ_p). κ_p is invariant under the reparameterization $s \mapsto cs$ for any $c > 0$, and under rescaling $q \mapsto aq$ for any $a > 0$.

Proof. Under $s \mapsto cs$: the new function is $\tilde{\phi}(\tilde{s}) = \phi(\tilde{s}/c)$, so $\tilde{\phi}'(\tilde{s}) = \phi'(\tilde{s}/c)/c$ and $\tilde{s}_p = cs_p$. Then $|\tilde{\phi}'(\tilde{s}_p) \tilde{s}_p / \tilde{\phi}(\tilde{s}_p)| = |\phi'(s_p) s_p / \phi(s_p)| = \kappa_p$. Under $q \mapsto aq$: replace ϕ by $a\phi$; then ϕ' scales by a and q_p scales by a , leaving κ_p unchanged. $\square$

This is the key advantage over the raw inverse sensitivity $1/\phi'(s)$: a simple rescaling of the parameter space cannot change κ_p .

3.3. Feasibility and uniqueness

Proposition 3.1 (Feasibility). Let $q_p = \phi(s_p)$ with $s_p \in S_p$, where ϕ is continuous and S_p is an interval. Define $m_p = \inf_{s_p \in S_p} \phi(s_p)$ and $M_p = \sup_{s_p \in S_p} \phi(s_p)$. Demand conservation Eq. (1) for path w requires

$$\sum_{p \in P_w} m_p \leq b_w \leq \sum_{p \in P_w} M_p. \quad (7)$$

Moreover: (a) if S_p is a closed bounded interval, $\phi(S_p) = [m_p, M_p]$ and any b_w in $[\sum m_p, \sum M_p]$ is achievable; (b) if S_p is semi-infinite with one attained endpoint, the achievable range is half-open; (c) if $S_p = \mathbb{R}$ and ϕ has limiting values m_p, M_p not attained at any finite s_p , the achievable range is the open interval $(\sum m_p, \sum M_p)$.

Proof. Summing $m_p \leq \phi(s_p) \leq M_p$ over P_w and using Eq. (1) gives necessity. The bounds in Eq. (7) are therefore necessary. Sufficiency: since ϕ is continuous on the interval S_p , the image $\phi(S_p)$ is an interval by the intermediate value theorem. The achievability cases follow from whether the infimum/supremum are attained: case (a) $S_p = [a, b]$, both endpoints attained; case (b) $S_p = [a, \infty)$, infimum attained, supremum is a limit; case (c) $S_p = \mathbb{R}$, neither limit attained (exponential with $m_p = 0, M_p = \infty$: any $b_w > 0$ is achievable, but $b_w = 0$ requires $s \rightarrow -\infty$). $\square$

Proposition 3.2 (Uniqueness). If ϕ is injective on S_p for every path (condition C4), then every feasible flow vector q has at most one preimage parameter vector s . Conversely, if ϕ fails C4, a single flow vector can have multiple preimages.

Proof. If $\phi(s) = q = \phi(s')$ and ϕ is injective componentwise, then $s_p = s'_p$ for every path, giving $s = s'$. Conversely, non-injectivity produces distinct $s_1 \neq s_2$ with $\phi(s_1) = \phi(s_2)$, yielding two parameter vectors for one flow. $\square$

3.4. Conditional equilibrium invariance

Theorem 3.1 (Equilibrium invariance — conditional). Let ϕ and ψ be two admissible substitutions with parameters s^* and r^* satisfying $\phi(s^*) = \psi(r^*) = q^*$. Then:

$$G(\phi(s^*)) = G(\psi(r^*)) = G(q^*), \quad (8)$$

and q^* is a Wardrop equilibrium under the original model if and only if the complementarity condition Eq. (5) holds for both (s^*, ϕ) and (r^*, ψ) . In particular, the link flow $f^* = Aq^*$ and equilibrium costs u^* are identical under both parameterizations.

Proof. Eq. (8) follows because both $f = Aq$ and $G(q) = A^T T(Aq)$ depend on the parameter only through $\phi(s^*) = q^*$, which is fixed. Every quantity in Eq. (4) is therefore determined by q^* alone. $\square$

Remark 1. Consequently, Theorem 1 is a *conditional* equivalence, not a universal one. It states that IF two substitutions have already produced the same flow q^* , THEN all equilibrium quantities agree. It does *not* assert that independently applied substitutions will converge to the same q^* . On the other hand, the equilibrium q^* is determined by the Wardrop conditions Eq. (4), so we say the choice of ϕ does not alter it. As a result, the significance of Theorem 1 is that it precisely identifies what reparameterization changes, this implies, the feasibility set, zero-flow representability, parameter uniqueness, as well as condition number κ_p , versus what it leaves invariant, justifying that equilibrium flow, link flow, in addition to costs remain unchanged.

4. Classification of admissible substitutions

Throughout this section, q and s denote a generic path flow and parameter.

Theorem 4.1 (Classification). Each of the following six substitutions is admissible in the sense of Definition 3.2 on the stated recommended domain. Parts (a), (b), (d), (f) additionally have the zero-flow property (Definition 3.3); parts (c) and (e) do not. Part (e) is infeasible when demand is below the number of available paths.

(a) *Cosine.* $q = \cos(s)$, recommended domain $[0, \pi/2]$, on which $0 \leq \cos(s) \leq 1$:

$$\phi'(s) = -\sin(s), \quad |\phi'(s)| = \sqrt{1 - q^2}, \quad s = \arccos(q), \quad s^0 = \pi/2. \quad (9)$$

The cosine relations are summarised in Eq. (9).

(b) *Sine.* $q = \sin(s)$, recommended domain $[0, \pi/2]$, on which $0 \leq \sin(s) \leq 1$:

$$\phi'(s) = \cos(s) = \sqrt{1 - q^2}, \quad s = \arcsin(q), \quad s^0 = 0. \quad (10)$$

The sine relations are summarised in Eq. (10).

(c) *Exponential.* $q = e^s$, domain $\mathbb{R}$, $e^s > 0$ for all $s \in \mathbb{R}$:

$$\phi'(s) = e^s = q, \quad s = \ln(q). \quad (11)$$

The exponential relations are summarised in Eq. (11). No finite s^0 exists; zero flow requires $s \rightarrow -\infty$ (C3 fails).

(d) *Hyperbolic sine.* $q = \sinh(s) = (e^s - e^{-s})/2$, recommended domain $[0, \infty)$:

$$s \geq 0 \Rightarrow \sinh(s) \geq 0, \quad \sinh(0) = 0; \quad \phi'(s) = \cosh(s) = \sqrt{1 + q^2}, \quad s = \operatorname{arsinh}(q), \quad s^0 = 0. \quad (12)$$

The hyperbolic-sine relations are summarised in Eq. (12).

(e) *Direct hyperbolic cosine.* $q = \cosh(s) = (e^s + e^{-s})/2$, domain $\mathbb{R}$, $\cosh(s) \geq 1$ for all s :

$$\sum_{p \in P_w} \cosh(s_p) \geq n_w \Rightarrow b_w \geq n_w. \quad (13)$$

The direct hyperbolic-cosine feasibility bound is given in Eq. (13). Infeasible if $b_w < n_w$; C3 fails (cosh never reaches zero).

(f) *Adjusted hyperbolic cosine.* $q = \cosh(s) - 1$, recommended domain $[0, \infty)$:

$$\cosh(s) - 1 \geq 0, \quad \cosh(0) - 1 = 0; \quad \phi'(s) = \sinh(s) = \sqrt{(q+1)^2 - 1}, \quad s = \operatorname{arcosh}(q+1), \quad s^0 = 0. \quad (14)$$

The shifted hyperbolic-cosine relations are summarised in Eq. (14). Injectivity requires $s \geq 0$ because cosh is even.

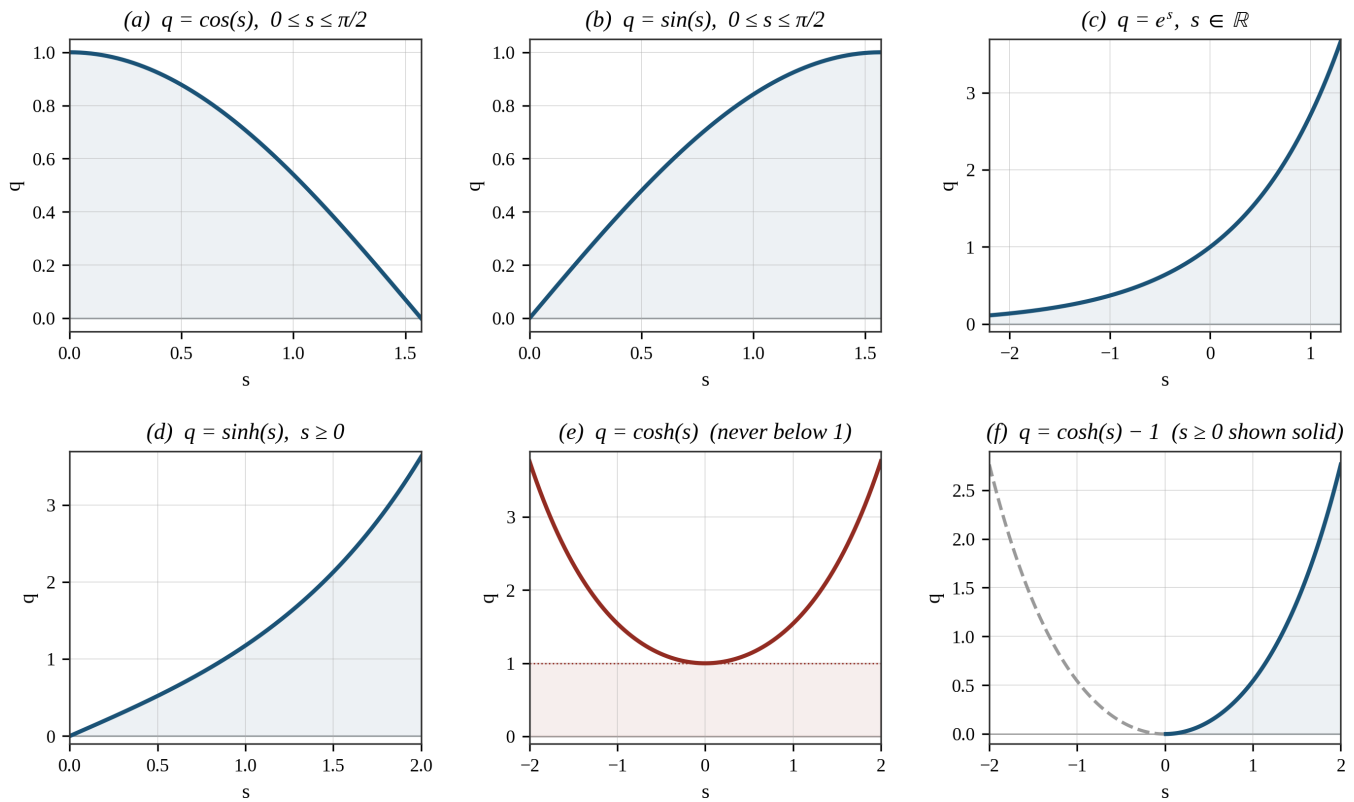


Figure 1: The six substitutions of Theorem 4.1 on their recommended domains. Blue shading shows the non-negative range; the dashed branch in panel (f) is excluded by the domain restriction $s \geq 0$. Panel (e) (red) is infeasible whenever demand $b_w < n_w$.

Proof of Theorem 2

Proof. (a)–(b): $|\cos(s)|, |\sin(s)| \leq 1$ everywhere; non-negativity on $[0, \pi/2]$ follows from the monotonicity of each on that interval. Injectivity holds because $\cos$ is strictly decreasing and $\sin$ strictly increasing on $[0, \pi/2]$. Derivatives and inverses are standard calculus. (c): $e^s > 0$ on $\mathbb{R}$; strictly increasing hence injective; derivative equals itself; inverse is $\ln$; no finite zero. (d): $\sinh$ is odd and strictly increasing, so non-negative and injective on $[0, \infty)$ with $\sinh(0) = 0$; $(\cosh(s))^2 - (\sinh(s))^2 = 1$ gives the derivative. (e)–(f): $\cosh(s) \geq 1$ with equality only at $s = 0$; summing over n_w paths gives Eq. (13), confirming infeasibility for small demand. Subtracting 1 shifts the minimum to zero (restoring C3) while preserving strict increase on $[0, \infty)$ (restoring C4 once the domain is restricted); the derivative identity follows as in (d). $\square$

Figure 1 displays the six substitutions on the recommended domains stated in Theorem 4.1. Panels (a)–(d) and (f) are non-negative and reach zero at a finite parameter; panel (e) is shown in red to highlight its infeasibility below $q = 1$.

5. Numerical illustration: four-node communication network

5.1. Network structure

Figure 2 shows a four-node communication network $D = (V, L)$ with node set $V = \{O, A, B, D\}$ (origin O , intermediate nodes A and B , destination D) and five directed links:

$$d_1 = (O \rightarrow A), d_2 = (O \rightarrow B), d_3 = (A \rightarrow D), d_4 = (B \rightarrow D), d_5 = (A \rightarrow B).$$

Three paths route one unit of demand ($b = 1$) from O to D : $P_1 = O \rightarrow A \rightarrow D$, $P_2 = O \rightarrow B \rightarrow D$, $P_3 = O \rightarrow A \rightarrow B \rightarrow D$.

The link/path incidence matrix is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad f = Aq = \begin{pmatrix} q_1 + q_3 \\ q_2 \\ q_1 \\ q_2 + q_3 \\ q_3 \end{pmatrix}.$$

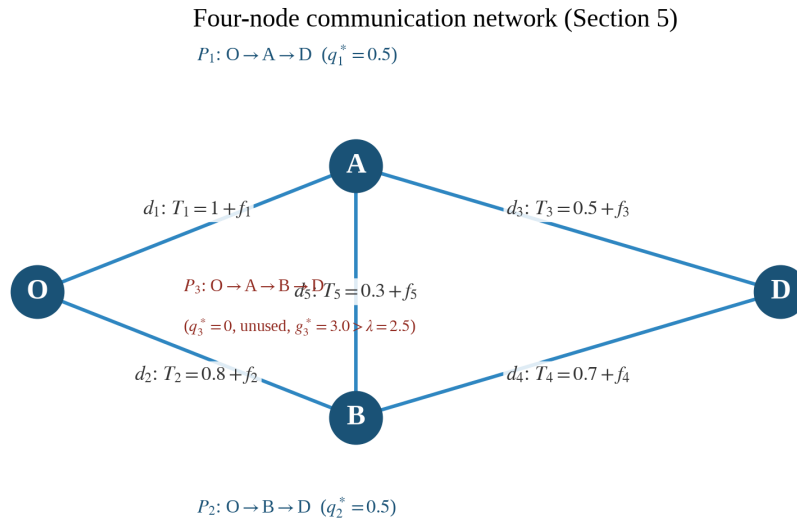


Figure 2: Four-node, five-link communication network for the numerical example. Path P_3 (red) is unused at equilibrium.

Table 1: Link flows, costs and path costs at equilibrium Eq. (19).

Link	f_d^*	$T_d(f_d^*)$	Path	$g_p(q^*)$	Status
d_1	0.5	1.5	P_1	2.5	Active ($q^* = 0.5$)
d_2	0.5	1.3	P_2	2.5	Active ($q^* = 0.5$)
d_3	0.5	1.0	P_3	3.0	Unused ($q^* = 0, g_3^* > \lambda$)
d_4	0.5	1.2			
d_5	0.0	0.3	λ	2.5	

The link cost functions are linear, BPR-type, and congestion-sensitive:

$$T_1 = 1 + f_1, \quad T_2 = 0.8 + f_2, \quad T_3 = 0.5 + f_3, \quad T_4 = 0.7 + f_4, \quad T_5 = 0.3 + f_5. \quad (15)$$

The path costs, derived from Eq. (3) and Eq. (15), are:

$$g_1(q) = T_1 + T_3 = 1.5 + 2q_1 + q_3, \quad (16)$$

$$g_2(q) = T_2 + T_4 = 1.5 + 2q_2 + q_3, \quad (17)$$

$$g_3(q) = T_1 + T_5 + T_4 = 2.0 + q_1 + q_2 + 3q_3. \quad (18)$$

Paths P_1 and P_3 share link d_1 ; paths P_2 and P_3 share link d_4 . These shared links create genuine path-cost interdependence through $f = Aq$.

5.2. Wardrop equilibrium

Using Eqs. (16), (17), and (18), we test for an interior equilibrium ($q_1, q_2, q_3 > 0$). $g_1 = g_2$ gives $q_1 = q_2$; $g_1 = g_3$ then yields $q_3 = -0.25 < 0$, contradicting non-negativity. Therefore path P_3 is unused at equilibrium ($q_3 = 0$). With $q_3 = 0$ and $q_1 = q_2$, the demand constraint $q_1 + q_2 = 1$ gives

$$q^* = (0.5, 0.5, 0), \quad \lambda = g_1(q^*) = 1.5 + 1 + 0 = 2.5. \quad (19)$$

Verification: $g_3(q^*) = 2.0 + 0.5 + 0.5 + 0 = 3.0 > 2.5 = \lambda$, confirming path P_3 is correctly unused (Wardrop condition Eq. (4) satisfied with $q_3^* = 0$ and $g_3^* > \lambda$). Table 1 collects the link flows and costs at equilibrium.

5.3. Parameter pre-images and the zero-flow property

Table 2 reports parameter pre-images for the active flow ($q^* = 0.5$) and the zero-flow value ($q^* = 0$ for the unused path P_3). The zero-flow column reveals directly which substitutions can represent unused paths at finite parameter values.

From this perspective, Figure 3 plots the five feasible substitutions across all three paths, illustrating Theorem 1 visually: the parameter vectors differ substantially, yet they all map back to the same equilibrium flow $q^* = (0.5, 0.5, 0)$. For path P_3 (unused), the exponential cannot place a bar because s^0 does not exist.

Table 2: Parameter pre-images for the equilibrium Eq. (19). s^0 is the zero-flow parameter; “—” means no finite zero-flow parameter exists.

Substitution	s at $q^* = 0.5$ (active)	s^0 at $q^* = 0$ (unused P_3)	Feasible?	C3 holds?
$q = \sin(s)$	0.52360	$s^0 = 0$	Yes	Yes
$q = \cos(s)$	1.04720	$s^0 = \pi/2 \approx 1.5708$	Yes	Yes
$q = e^s$	-0.69315	—	Yes (flows > 0)	No
$q = \sinh(s)$	0.48121	$s^0 = 0$	Yes ($s \geq 0$)	Yes
$q = \cosh(s)$	not real	—	No	No
$q = \cosh(s) - 1$	0.96242	$s^0 = 0$	Yes ($s \geq 0$)	Yes

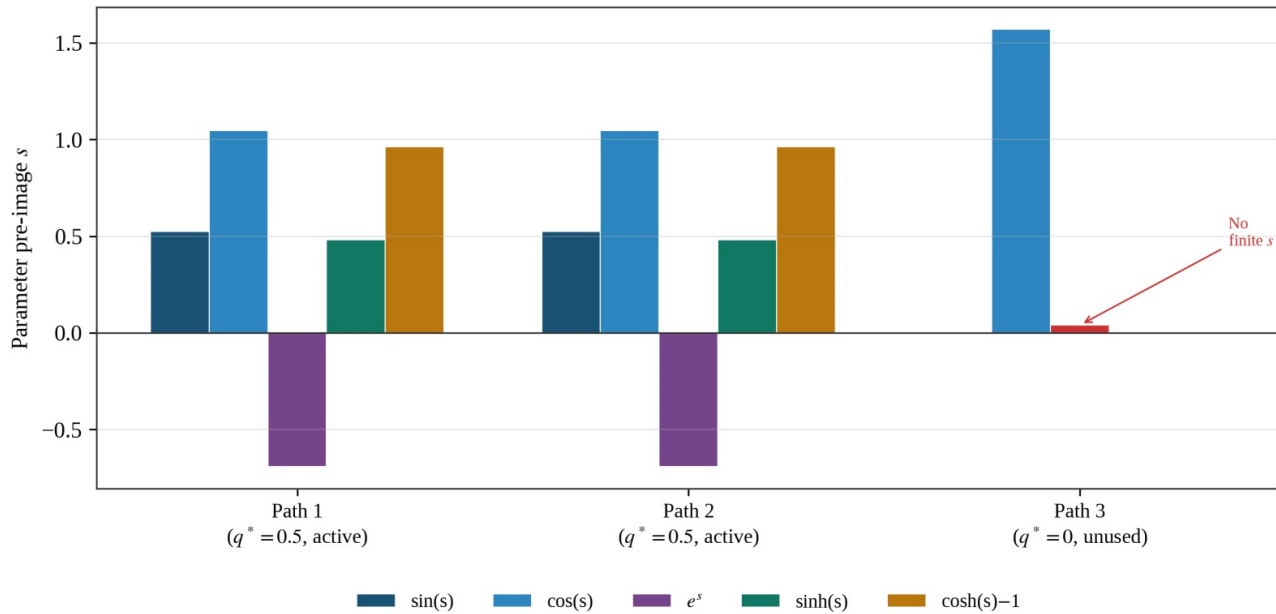


Figure 3: Parameter pre-images across all three paths (Table 2). For the unused path P_3 (rightmost group), the exponential (red arrow) has no finite zero-flow parameter, while all others achieve s^0 at a finite value.

Table 3: Dimensionless condition number $\kappa_p = |\phi'(s_p) s_p / q_p|$ at $q^* = 0.5$.

Substitution	s_p	$\phi'(s_p)$	κ_p
$q = \sin(s)$	0.52360	$\cos(s) = 0.8660$	0.907
$q = \cos(s)$	1.04720	$ \sin(s) = 0.8660$	1.814
$q = e^s$	-0.69315	$e^s = 0.5$	0.693
$q = \sinh(s)$	0.48121	$\cosh(s) = 1.1180$	1.076
$q = \cosh(s) - 1$	0.96242	$\sinh(s) = 1.1180$	2.152

5.4. Dimensionless condition numbers

Table 3 reports the dimensionless condition number $\kappa_p = |\phi'(s_p) s_p / q_p|$ for the five feasible substitutions at the active flow $q^* = 0.5$. By Lemma 3.1, these values are invariant under rescaling of s as well as q .

Figure 4 displays these values. The dotted reference line at $\kappa_p = 1$ marks unit relative sensitivity. The adjusted hyperbolic cosine has the highest κ_p (2.152), meaning a 1% relative change in s_p produces a 2.152% relative change in q_p . The exponential is the most stable in relative terms ($\kappa_p = 0.693$), despite having the largest raw sensitivity $1/\phi'(s) = 2.0$: the contrast arises because its parameter value $s = -0.693$ is small in magnitude. This distinction is not captured by the raw measure and underscores why scale invariance matters. At the same time, the $\cosh(s) - 1$ substitution yields $\kappa_p = 2.152$, justifying that greater numerical care is required when applying it near the zero-flow boundary.

6. Comparative results

Table 4 summarises the admissibility classification established in Theorem 4.1.

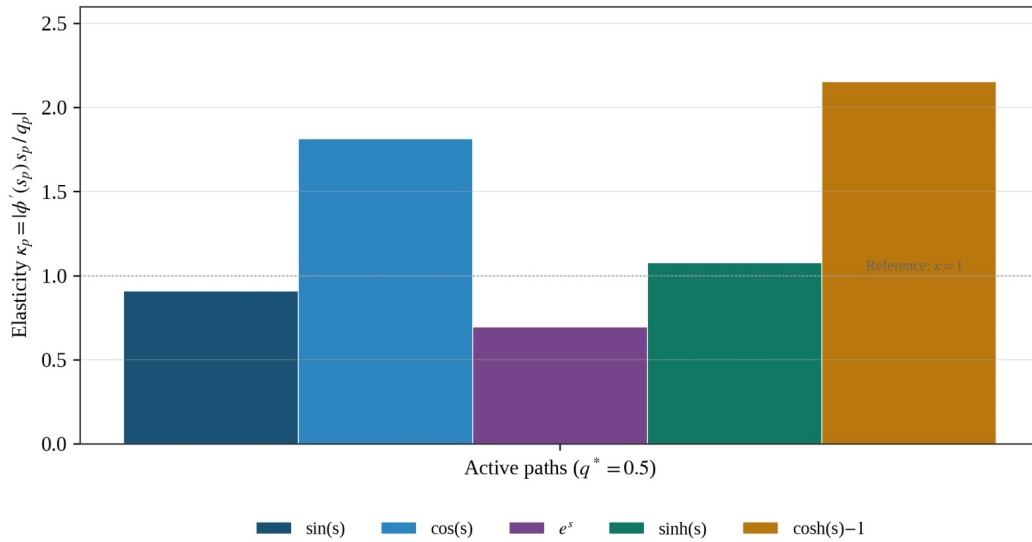


Figure 4: Dimensionless condition number κ_p at $q^* = 0.5$ (Table 3). The dotted line marks $\kappa_p = 1$. Unlike the raw $1/\phi'(s)$, κ_p is invariant under rescaling of the parameter s or the flow q (Lemma 3.1).

Table 4: Feasibility and structural comparison of the six substitutions (Theorem 4.1).

Substitution	Unrestricted range	Recommended domain	C3 (zero flow)	C4 (injective)	Main limitation
$\cos(s)$	$[-1, 1]$	$[0, \pi/2]$	Yes	Yes	Bounded; decreasing in s
$\sin(s)$	$[-1, 1]$	$[0, \pi/2]$	Yes	Yes	Bounded; needs scaling for $b_w > \sum c_p$
e^s	$(0, \infty)$	$\mathbb{R}$	No	Yes	No finite zero-flow parameter
$\sinh(s)$	$\mathbb{R}$	$[0, \infty)$	Yes	Yes	Needs domain restriction
$\cosh(s)$	$[1, \infty)$	$\mathbb{R}$	No	No	Infeasible if $b_w < n_w$
$\cosh(s) - 1$	$[0, \infty)$	$[0, \infty)$	Yes	Yes	Needs $s \geq 0$ for uniqueness

Corollary 6.1 (Choice of substitution). For an OD pair with n_w paths, per-path capacities $c_p > 0$, and demand $b_w > 0$:

- (i) The scaled substitutions $q = c_p \phi(s)$ with $\phi \in \{\cos, \sin\}$ are admissible with the zero-flow property if and only if $0 \leq b_w \leq \sum_{p \in P_w} c_p$, since each path carries at most c_p units and minimum 0 (from C3).
- (ii) For any $b_w > 0$, e^s is admissible without the zero-flow property; it is suitable when all equilibrium paths are known to be active.
- (iii) For any $b_w > 0$, $\sinh(s)$ and $\cosh(s) - 1$ are admissible with the zero-flow property; $\cosh(s) - 1$ requires $s \geq 0$ for parameter uniqueness.
- (iv) $\cosh(s)$ is admissible without C3 and is infeasible if $b_w < n_w$.

Proof. (i) The scaled image $c_p \phi(S_p) = [0, c_p]$ for $\phi \in \{\cos, \sin\}$ on $[0, \pi/2]$; by Proposition 1(a), demand conservation is achievable iff $0 \leq b_w \leq \sum_{p \in P_w} c_p$. The original statement used n_w as the upper bound, which assumed $c_p = 1$ for all p and is corrected here to $\sum c_p$. (ii)–(iv) restate columns of Table 4. $\square$

7. Conclusion

We presented a rigorous comparative treatment of six functional substitutions for path flow in a Wardrop network equilibrium model. Theorem 3.1, stated with explicit scope, shows that two substitutions producing the same flow yield identical equilibrium quantities; it does not claim universal equivalence. Theorem 4.1 classifies the six substitutions by their admissibility properties, zero-flow property, and scale-invariant condition number κ_p . The four-node network example in Section 5 demonstrates these properties concretely through an unused path. Representing this path requires a finite zero-flow parameter value, a condition that neither the exponential nor the direct hyperbolic cosine satisfies. The dimensionless condition numbers κ_p give a scale-invariant ranking that differs substantially from the naive raw-sensitivity ordering. Future work may extend the framework to dynamic, multi-commodity, or capacity-constrained equilibrium models.

Data availability

This is a theoretical study; no external datasets were generated or analysed. All numerical values reported in Section 5, Tables 1–4, and Appendix A are derived analytically from the network parameters and cost functions stated in the manuscript and can be reproduced in full from Eqs. (15)–(19) and the closed-form inverses in Theorem 4.1. The complete reproduction script (Python) is available from the corresponding author on reasonable request.

Declaration of competing interest

The authors declare no competing interests.

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Appendix A. Derivation of the network equilibrium

This appendix gives the algebra for Section 5.2. Assume all three paths active: $g_1 = g_2$ gives $q_1 = q_2$; $g_1 = g_3$ then gives $q_3 = (1.5 - 2.0)/(3 - 1) = -0.25 < 0$. Hence $q_3 = 0$. With $q_3 = 0$: $g_1 = g_2 = 1.5 + 2q_1$, $q_1 + q_2 = 1$, $q_1 = q_2$ gives $q_1 = q_2 = 0.5$ and $\lambda = 2.5$. Unused-path check: $g_3(0.5, 0.5, 0) = 2.0 + 0.5 + 0.5 = 3.0 > 2.5 = \lambda$. $\square$

Appendix B. Summary of inverse transformations

For $q = \sin(s)$, the inverse is $s = \arcsin(q)$ for $0 \leq q \leq 1$, and C3 holds. For $q = \cos(s)$, the inverse is $s = \arccos(q)$ for $0 \leq q \leq 1$, and C3 holds. For $q = e^s$, the inverse is $s = \ln(q)$ for $q > 0$, and C3 does not hold. For $q = \sinh(s)$, the inverse is $s = \operatorname{arsinh}(q)$ for $q \geq 0$ with $s \geq 0$, and C3 holds. For $q = \cosh(s)$, the inverse is $s = \operatorname{arcosh}(q)$ for $q \geq 1$, and C3 does not hold. For $q = \cosh(s) - 1$, the inverse is $s = \operatorname{arcosh}(q + 1)$ for $q \geq 0$ with $s \geq 0$, and C3 holds.